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A FINITE ELEMENT MODEL FOR ACOUSTIC RADIATION

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In this article, an extension of the finite element technique to the analysis of the acoustic radiation is presented. In the proposed approach, the acoustic domain is split into two parts by an arbitrary artificial boundary enclosing the radiating surface. Then the unbounded medium is discretized with finite elements consisting of only one or two rows in the radial direction up to the artificial boundary. To represent the farfield behaviour, the constraint equation specified on the artificial boundary is formulated with a more straightforward boundary integral equation, in which the source surface coincides with the radiating surface discretized with boundary elements. To ensure the uniqueness of the numerical solution, the composite Helmholtz integral equation proposed by Burton and Miller was adopted. Due to the avoidance of singularity problems inherent in the boundary element formulation, this method is very efficient and easy to implement in an isoparametric element environment. In numerical experiments involving spherical and cubical radiators, it has been demonstrated that the proposed method eliminates the difficulties when the FEM handles the exterior acoustics. It should be noted that the method can be extended to the computation on other branches of the classical field theory.

1. INTRODUCTION

The determination of the acoustic field radiated by an arbitrary shaped vibrating structure immersed in an infinite homogeneous acoustic medium is of considerable interest in many areas including underwater acoustics and aeronautics.

Of various numerical methods used in mechanical engineering, the Finite Element Method (FEM) is perhaps the most popular. The field of acoustics can be solved entirely through finite element modelling, which started in the mid-1960s with a paper by Melvyn [1]. This technique gives satisfactory solutions in an enclosed space [2–5]. However, the FEM is best suited to finite rather than infinite geometries. In the case of exterior acoustics, where the fluid occupies an unbounded domain, the vast amount of data to be handled makes it difficult to apply the FEM. A classical finite element model of the problem requires that the mesh of elements be extended sufficiently far away from the vibrating surface so that the conditions imposed on the boundary of the mesh do not have an appreciative effect on the solution in the vicinity of the surface. In addition, special care has to be taken to insure the Sommerfeld radiation condition at infinity [6, 7], which enforces that no energy is radiated from infinity towards the obstacle. This radiation condition can be incorporated in certain analytical solutions and in particular the DtN (Dirichlet-to-Neumann) boundary conditions [8–11]. The DtN conditions have infinite series, and hence must be truncated after a finite number of terms. The unbounded medium can also be modelled approximately with infinite elements [12–17]. The element domain is extended to infinity, using as a basis any original finite element. Decay functions

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representing the wave propagation towards infinity are used as the shape functions. Several excellent books [18–20] give details of infinite element concepts. However, the infinite-element method is not exact in the finite element sense [20].

The Boundary Integral Equation Method (BIEM) has long been an effective numerical technique for acoustic problems [21–24]. The main feature of this method is that it can handle the Sommerfeld radiation condition automatically. Although the BIEM is regarded as the most powerful procedure for modelling the unbounded medium in many areas of engineering, it needs quite sophisticated mathematics. One of the main difficulties in the BIEM for acoustic problems is that the Helmholtz integral equation has some singular integrals of high order. Extensive work has been done to address this shortcoming [25–28]. Unfortunately, the end result is always to increase the complexity, and thus the extent, of the computations.

The search to find a simpler, more straightforward computational method that circumvents the above difficulties was the motivation behind this study. The proposed scheme is based on finite element and boundary element concepts. It is in some ways similar to the work of Givoli and Cohen [10], in which the non-reflecting boundary conditions based on Kirchhoff-type formulae were used to solve time-domain problems. In the proposed approach, the acoustic domain is split into two parts by an arbitrary artificial boundary enclosing the radiating surface. Then the unbounded medium is discretized with finite elements consisting of one or two rows only in the radial direction up to the artificial boundary (Figure 1). To represent the farfield behaviour, the constraint equation specified on the artificial boundary is formulated with a more straightforward boundary integral equation, in which the source surface coincides with the radiating surface discretized with boundary elements. Thus, the Sommerfeld radiation condition at the farfield boundary is automatically satisfied. Note that in this method, unlike the existing combined finite element and boundary element subregion methods [29], the field points of the kernels of the integrals are restricted to exterior regions well away from radiating boundaries. So the difficulty of a singular integral can be avoided. Although this coupling strategy is very simple, it is not widely used. Lenoir and Jami [30] have developed a variational formulation of the approach, and applied it to two-dimensional hydrodynamics. However, this method fails to provide a unique solution at certain characteristic frequencies that depend on the shape of the body, the boundary conditions imposed, and the shape of the artificial

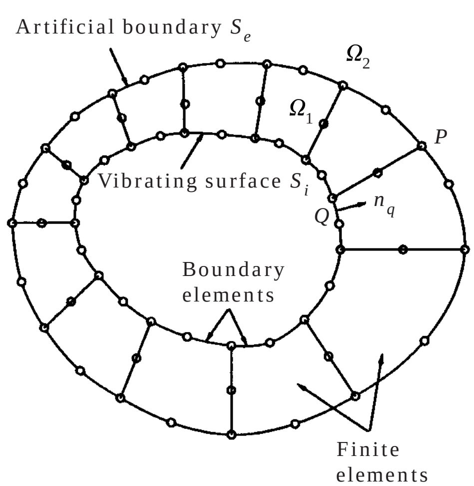

Figure 1. The partitioned domain.

boundary. In this paper, a modified formulation uses the composite Helmholtz integral equation proposed by Burton and Miller [25] to ensure the uniqueness of the numerical solution at all frequencies. It is shown in the applications that this method is very efficient and easy to implement in an isoparametric element environment.

2. FINITE ELEMENT FORMULATION

The formulation of the finite element method for time harmonic acoustic wave propagation is well documented [2, 31]. The method produces a matrix equation of the form

$$[k^2[\mathbf{M}] - [\mathbf{K}]]\{\phi\} = \{\mathbf{f}\} \tag{1}$$

where k is the wavenumber, $[\mathbf{M}]$ is the acoustic mass matrix, $[\mathbf{K}]$ is the acoustic stiffness matrix, $\{\phi\}$ is the vector of acoustic potential function, and $\{\mathbf{f}\}$ is the acoustic forcing vector.

The formulation used in equation (1) is

$$[\mathbf{M}]_e = \int_V \{\mathbf{N}\}\{\mathbf{N}\}^T \,\mathrm{d}V, \qquad [\mathbf{K}]_e = \int_V [\nabla\mathbf{N}][\nabla\mathbf{N}]^T \,\mathrm{d}V \tag{2, 3}$$

and

$$\{\mathbf{f}\}_e = \mathrm{i}\omega\rho \int_S \{\mathbf{N}_S\}\{\mathbf{N}_S\}^T\{\mathbf{v}_n\}\,\mathrm{d}S = -\int_S \{\mathbf{N}_S\}\{\mathbf{N}_S\}^T\left\{\frac{\partial\phi}{\partial n}\right\}\mathrm{d}S, \tag{4}$$

where ρ is the fluid density, ω is the angular frequency, $\mathbf{v}_n$ is the normal particle velocity, $\partial\phi/\partial n = -i\omega\rho\mathbf{v}_n$ is the acoustic potential gradient, $\{\mathbf{N}\}$ is the vector of shape functions, and $[\nabla\mathbf{N}]$ is a matrix of shape function derivatives of m-node element of the form

$$[\nabla\mathbf{N}] = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial z} \\ \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial z} \\ \vdots & \vdots & \vdots \\ \dfrac{\partial N_m}{\partial x} & \dfrac{\partial N_m}{\partial y} & \dfrac{\partial N_m}{\partial z} \end{bmatrix}. \tag{5}$$

The e subscript indicates that the matrices are formulated for a single element. Note that $\{\mathbf{f}\}$ is evaluated using boundary integrals rather than volume integrals, and $\{\mathbf{N}_s\}$ used in equation (4) denotes the vector of shape function for the surface of integration.

Equation (1) can be represented by the following matrix equation,

$$[\mathbf{A}]\{\phi\} = [\mathbf{B}]\left\{\frac{\partial\phi}{\partial n}\right\}, \tag{6}$$

where $[\mathbf{A}] = k^2[\mathbf{M}] - [\mathbf{K}]$, and $[\mathbf{B}] = -\int_S \{\mathbf{N}_S\}\{\mathbf{N}_S\}^T\,\mathrm{d}S$.

The interior boundary of the finite element region coincides with the radiating surface and its exterior boundary with the artificial boundary. Partitioning into the nodes lying on the interior (index i) and exterior (index e) boundaries, and the remaining nodes (index r) yields

$$[\mathbf{A}_i \ \ \mathbf{A}_r \ \ \mathbf{A}_e]\begin{Bmatrix}\phi_i\\ \phi_r\\ \phi_e\end{Bmatrix} = [\mathbf{B}_i \ \ \mathbf{B}_e]\begin{Bmatrix}(\partial\phi/\partial n)_i\\ (\partial\phi/\partial n)_e\end{Bmatrix}. \tag{7}$$

3. BOUNDARY INTEGRAL FORMULATION FOR CONSTRAINT EQUATIONS

To satisfy the Sommerfeld radiation condition, a boundary condition must be introduced on the artificial boundary. It can be formulated straightforwardly with the following classical Helmholtz integral equation:

$$4\pi\phi(P) = \int_{S_i}\left[\phi(Q)\frac{\partial G_k(P,Q)}{\partial n_q} - G_k(P,Q)\frac{\partial\phi(Q)}{\partial n_q}\right]\mathrm{d}S_i, \tag{8}$$

which is valid for an acoustic medium exterior to a vibrating surface S_i, k is the wavenumber; ϕ is the acoustic potential function; $\partial/\partial n_q$ represents an outward normal derivative with respect to the body S; and the function G_k is the free-space Green's function

$$G_k(P,Q) = \frac{\mathrm{e}^{-\mathrm{i}kR(P,Q)}}{R(P,Q)}, \tag{9}$$

where $R(P,Q) = |P - Q|$ is the physical distance between points P and Q. The required integrations can be carried out using the standard Gaussian quadrature because, with the point P not on the radiating surface, none of the kernel functions is singular.

For numerical implementation, the matrix form of equation (8) may be expressed as

$$\{\phi(P)\} = [\mathbf{C}]\{\phi(Q)\} - [\mathbf{D}]\left\{\frac{\partial\phi(Q)}{\partial n_q}\right\}. \tag{10}$$

Because the vibrating surface coincides with the interior boundary (index i) of the finite element region and the field points are located on the artificial boundary (index e), equation (10) can be written as

$$\{\phi_e\} = [\mathbf{C}]\{\phi_i\} - [\mathbf{D}]\left\{\left(\frac{\partial\phi}{\partial n}\right)_i\right\}. \tag{11}$$

To eliminate the problem of uniqueness at critical wavenumber, one can employ the Burton and Miller formulation [25] which uses a linear combination of the Helmholtz integral equation and its normal derivative as follows:

$$4\pi\left[\phi(P) + \alpha\frac{\partial\phi(P)}{\partial n_p}\right] = \int_{S_i}\left[\phi(Q)\frac{\partial G_k(P,Q)}{\partial n_q} - G_k(P,Q)\frac{\partial\phi(Q)}{\partial n_q}\right]\mathrm{d}S_i,$$

$$+\,\alpha\int_{S_i}\left[\phi(Q)\frac{\partial^2 G_k(P,Q)}{\partial n_p\,\partial n_q} - \frac{\partial G_k(P,Q)}{\partial n_p}\frac{\partial\phi(Q)}{\partial n_q}\right]\mathrm{d}S_i, \tag{12}$$

where α is a coupling constant. Similarly, the matrix form of equation (12) may be expressed as

$$\{\phi_e\} + \left\{\alpha\left(\frac{\partial\phi}{\partial n}\right)_e\right\} = [\mathbf{C}^*]\{\phi_i\} - [\mathbf{D}^*]\left\{\left(\frac{\partial\phi}{\partial n}\right)_i\right\}. \tag{13}$$

4. SOLUTION OF THE SYSTEM

Now, combining equations (7) and (11) yields

$$\begin{bmatrix} \mathbf{A}_i & \mathbf{A}_r & \mathbf{A}_e & -\mathbf{B}_e \\ -\mathbf{C} & 0 & \mathbf{I} & 0 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \phi_r \\ \phi_e \\ \left(\frac{\partial\phi}{\partial n}\right)_e \end{Bmatrix} = \begin{bmatrix} \mathbf{B}_i \\ -\mathbf{D} \end{bmatrix} \left\{\left(\frac{\partial\phi}{\partial n}\right)_i\right\}, \tag{14}$$

and combining equations (7) and (13) yields

$$\begin{bmatrix} \mathbf{A}_i & \mathbf{A}_r & \mathbf{A}_e & -\mathbf{B}_e \\ -\mathbf{C}^* & 0 & \mathbf{I} & \alpha\mathbf{I} \end{bmatrix} \begin{Bmatrix} \phi_i \\ \phi_r \\ \phi_e \\ \left(\frac{\partial\phi}{\partial n}\right)_e \end{Bmatrix} = \begin{bmatrix} \mathbf{B}_i \\ -\mathbf{D}^* \end{bmatrix} \left\{\left(\frac{\partial\phi}{\partial n}\right)_i\right\}, \tag{15}$$

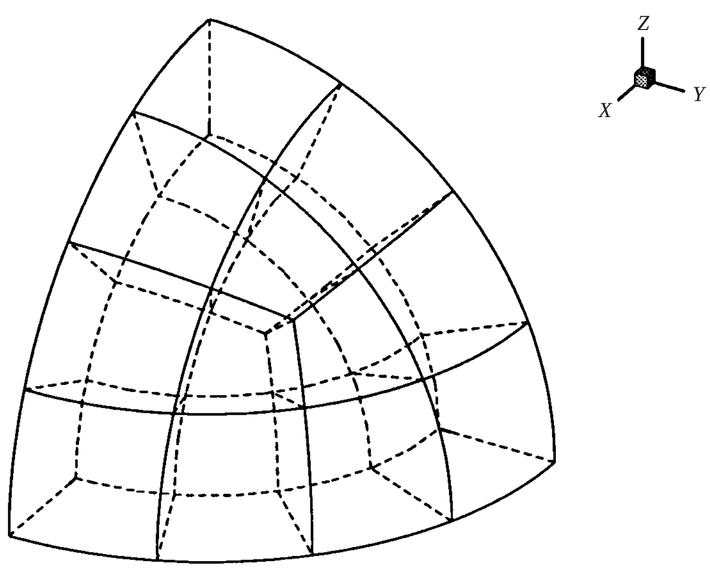

Figure 2. Vicinity of one octant of a sphere modelled using twelve 20-noded finite elements and twelve 8-noded boundary elements.

Table 1

Results for a pulsating sphere

k	Analytical solution	Equation (14)	Equation (15)
1	0·5403 − 0·8415i	−0·5400 − 0·8417i	−0·5400 − 0·8416i
2	−0·4161 − 0·9093i	−0·4168 − 0·9120i	−0·4155 − 0·9101i
3	−0·9900 − 0·1411i	−0·9915 − 0·1498i	−0·9914 − 0·1477i
π	−1·0000 + 0·0000i	−0·5402 − 1·7132i	−1·0024 − 0·0092i
4	−0·6536 + 0·7568i	−0·6428 + 0·7816i	−0·6506 + 0·8025i
5	0·2837 + 0·9589i	0·3040 + 0·9742i	0·3033 + 0·8942i
6	0·9602 + 0·2794i	0·9781 + 0·2641i	0·9536 + 0·2969i
2π	1·0000 + 0·0000i	0·7241 + 4·6575i	0·9987 + 0·0204i

where **I** is the identity or unit matrix.

One can reorder equations (14) and (15) with all the unknowns ϕ_i and $(\partial\phi/\partial n)_i$ on the left side and a vector on the right side obtained by multiplying matrix elements by the known values of acoustic potential and potential gradient. Once the boundary values of the acoustic potential and its gradient have been computed at all the nodal points on the radiating surface, it is a straightforward matter to calculate the exterior potential at any desired point using equation (8).

5. NUMERICAL TESTS

In order to confirm the validity of the proposed formulation, we examine two test cases.

Radiation from a uniformly pulsating sphere of unit radius is a basic problem of acoustics. The surface velocity is given by

$$\frac{\partial\phi}{\partial n} = \frac{(-1 - \mathrm{i}kr)\,\mathrm{e}^{-\mathrm{i}kr}}{r^2}, \tag{16}$$

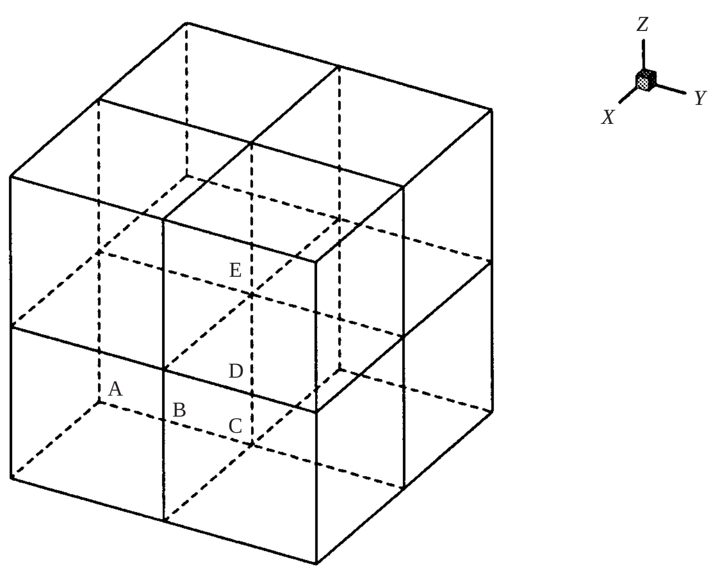

Figure 3. Vicinity of one-eight of a cube modelled using seven 20-noded finite elements and three 8-noded boundary elements.

where k is the wavenumber and r is the radius of sphere. The corresponding dimensionless surface potential is

$$\phi = \frac{e^{-ikr}}{r}. \tag{17}$$

TABLE 2

Results at point A for a pulsating cube

k	Analytical solution	Computed result
0·5	1·9378 − 0·4948i	1·9012 − 0·4883i
1·0	1·7552 − 0·9589i	1·7215 − 0·9480i
1·5	1·4634 − 1·3633i	1·4316 − 1·3495i
2·0	1·0806 − 1·6829i	1·0474 − 1·6651i
2·5	0·6306 − 1·8980i	0·6006 − 1·8566i
3·0	0·1415 − 1·9950i	0·1392 − 2·0146i

TABLE 3

Results at point B for a pulsating cube

k	Analytical solution	Computed result
0·5	1·7194 − 0·4935i	1·6912 − 0·4870i
1·0	1·5166 − 0·9487i	1·4918 − 0·9381i
1·5	1·1960 − 1·3303i	1·1741 − 1·3173i
2·0	0·7825 − 1·6086i	0·7606 − 1·5943i
2·5	0·3084 − 1·7621i	0·2809 − 1·7275i
3·0	−0·1897 − 1·7788i	−0·1840 − 1·8117i

TABLE 4

Results at point C for a pulsating cube

k	Analytical solution	Computed result
0·5	1·3267 − 0·4897i	1·3140 − 0·4832i
1·0	1·0752 − 0·9187i	1·0663 − 0·9080i
1·5	0·6906 − 1·2342i	0·6853 − 1·2207i
2·0	0·2205 − 1·3969i	0·2172 − 1·3798i
2·5	−0·2768 − 1·3869i	−0·2893 − 1·3493i
3·0	−0·7398 − 1·2053i	−0·7152 − 1·2297i

TABLE 5

Results at point D for a pulsating cube

k	Analytical solution	Computed result
0·5	1·2407 − 0·4884i	1·2273 − 0·4819i
1·0	0·9756 − 0·9089i	0·9664 − 0·8984i
1·5	0·5749 − 1·2030i	0·5698 − 1·1906i
2·0	0·0943 − 1·3300i	0·0913 − 1·3159i
2·5	−0·3994 − 1·2721i	−0·4128 − 1·2408i
3·0	−0·8376 − 1·0374i	−0·8186 − 1·0610i

TABLE 6

Results at point E for a pulsating cube

k	Analytical solution	Computed result
0·5	1·0481 − 0·4845i	1·0397 − 0·4781i
1·0	0·7481 − 0·8796i	0·7440 − 0·8693i
1·5	0·3100 − 1·1123i	0·3103 − 1·1000i
2·0	−0·1854 − 1·1397i	−0·1817 − 1·1251i
2·5	−0·6465 − 0·9567i	−0·6483 − 0·9467i
3·0	−0·9883 − 0·5972i	−0·9612 − 0·6183i

A spherical artificial boundary of radius 2·0 was selected. Figure 2 shows the element definition for the vicinity of an octant of the radiating surface used in this test case. Due to the symmetric property of the boundary condition, only a quarter of the acoustic domain was modelled. With the boundary integral formulation, symmetry was obtained by reflecting each integration point to the other parts. The finite element region was discretized by one layer of twenty-four 20-noded brick elements, and the radiating surface had twenty-four 8-noded quadrilateral boundary elements. A 4-point Gaussian quadrature formula was used for the numerical integration over each element. The computed results from equations (14) and (15) are given in Table 1. It can be seen that consistently good results were obtained with equation (15) whilst equation (14) failed at the characteristic wavenumbers.

As a truly three-dimensional problem, the second case tests the radiation by a pulsating cube of unit length. The normal velocity on the cubical surface is produced by a point source of spherical dilatation wave with unit intensity located at the cubical center. A cubical artificial boundary is introduced of length 2. Because of the symmetric boundary condition, one-quarter of the computational domain was discretized by fourteen 20-noded brick elements and six 8-noded quadrilateral boundary elements. Figure 3 shows the element definitions for one-eight of the acoustic field. The computed results at selected points shown in Figure 3 are tabulated in Tables 2–6. Acceptable results were obtained once again, the maximum relative error being 3·0% for the corner node at $k = 3$.

6. CONCLUSIONS

This paper has presented a finite element based approach with constraint equation formulated by boundary integral equation to the exterior acoustic radiation. The main feature of the proposed method is that it can handle the Sommerfeld radiation condition automatically, and the implementation is very straightforward because of the absence of integral singularities and sophisticated mathematics. By comparing the numerical results with the analytical solutions, it has been demonstrated that the proposed method eliminates the difficulties when the FEM handles the exterior acoustics. Theoretically, this technique can also be applied to the problems in bounded interior domains. It should be noted that the method can be extended to the computation on other branches of the classical field theory, and the implementation in a existing finite element code is quite easy.

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